PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 2
PRACE MATEMATYCZNE I, 1969

JANUSZ MATKOWSKI

On some properties of solutions of a functional equation

In the present paper we are concerned with the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{1}$$

where $f(x)$ and $h(x, y)$ are known real-valued functions of real variables and $\varphi(x)$ is unknown.

We assume the following hypotheses:

(I) $f(x)$ is defined and continuous in an interval $\langle 0, a)$ and

$$0 < f(x) < x \quad \text{for} \quad x \epsilon (0, a); \tag{2}$$

(II) $h(x, y)$ is defined in a domain Ω such that $(0, 0) \epsilon \Omega$; moreover, for every $x \epsilon \langle 0, a)$, the set $\Omega_x \overset{\text{def}}{=} \{y : (x, y) \epsilon \Omega\}$ is a nonempty open interval and $\Lambda_x \subset \Omega_x$, where $\Lambda_x \overset{\text{def}}{=} h(x, \Omega_{f(x)})$;

(III) $h(x, 0)$ is continuous at $x=0$, $h(0, 0)=0$ and

$$|h(x, y_1) - h(x, y_2)| \leqslant \upsilon |y_1 - y_2|, \quad 0 < \upsilon < 1, \tag{3}$$

holds in a neighbourhood of the point $(0, 0)$.

We shall prove the following

THEOREM. *Let hypotheses (I)—(III) be fulfilled. Then equation* (1) *has exactly one solution φ defined in $\langle 0, a)$, continuous at $x=0$ and such that $\varphi(0)=0$.*

1°. *If, moreover, $f(x)$ is strictly increasing in $\langle 0, a)$ and $h(x, y)$ is increasing with respect to either variable in Ω, then φ is increasing in $\langle 0, a)$ (if, for every fixed y, $h(x, y)$ is a strictly increasing function of x, then φ is strictly increasing in $\langle 0, a)$).*

2°. *If, besides* (I)—(III) *and hypotheses of* 1°, *we assume that $f(x)$ is convex in $\langle 0, a)$, Ω is a convex domain and $h(x, y)$ is a convex function of two variables in Ω, then φ is convex in $\langle 0, a)$.*

Proof. We choose a $c>0$ and a $d>0$ such that (3) holds in the set

$$D = \{(x, y) : 0 \leqslant x \leqslant c, \ -d \leqslant y \leqslant d\}.$$

By (III) we may assume that c has been chosen in such a manner that

$$|h(x, 0)| \leqslant (1-\upsilon) d \quad \text{for} \quad x \epsilon \langle 0, c\rangle. \tag{4}$$

Let F be the set of functions φ defined in $\langle 0, c \rangle$, continuous at $x=0$ and such that

(5) $$\varphi(0)=0\,; \qquad |\varphi(x)|\leqslant d \quad \text{for} \quad x \in \langle 0, c \rangle.$$

The set F with the metric

$$\varrho(\varphi_1, \varphi_2) = \sup_{\langle 0,c \rangle} |\varphi_1(x)-\varphi_2(x)|$$

is a complete metric space. We define the transform

(6) $$\psi(x) = h(x, \varphi[f(x)])$$

for $\varphi \in F$. It follows from (2) that

(7) $$|\varphi[f(x)]|\leqslant d \quad \text{for} \quad \varphi \in F \quad \text{and} \quad x \in \langle 0, c \rangle.$$

Now, from (3) and (4) we get

(8) $$|\psi(x)| = |h(x, \varphi[f(x)])| \leqslant |h(x, \varphi[f(x)]) - h(x, 0)| + |h(x, 0)| \leqslant \upsilon |\varphi[f(x)]| + |h(x, 0)|.$$

Hence and from (2) we see that $\lim\limits_{x\to 0+} \psi(x)=0$. Since $\psi(0)=h(0, \varphi(0))=h(0, 0)=0$, $\psi(x)$ is continuous at $x=0$. From (8) and (7) we obtain

$$|\psi(x)|\leqslant \upsilon d + (1-\upsilon) d = d.$$

This proves that (6) transforms F into itself. Further, we have in virtue of (3) for

$$\psi_1(x) = h(x, \varphi_1[f(x)]), \qquad \psi_2(x) = h(x, \varphi_2[f(x)]), \qquad \varphi_1, \varphi_2 \in F,$$

$$|\psi_1(x)-\psi_2(x)| = |h(x, \varphi_1[f(x)]) - h(x, \varphi_2[f(x)])| \leqslant \upsilon |\varphi_1[f(x)] - \varphi_2[f(x)]|,$$

whence

$$\varrho(\psi_1, \psi_2) \leqslant \upsilon \varrho(\varphi_1, \varphi_2),$$

i.e., (6) is a contraction map. On account of BANACH's theorem there exists exactly one solution $\varphi \in F$ of equation (1). This solution has a unique extension onto the whole interval $\langle 0, a)$ (cf. [1], p. 70, Theorem 3.2).

REMARK 1. In the book [1] Theorem 3.2 has been proved under the assumption of the continuity of $h(x, y)$ in Ω but Theorem 3.2 will remain valid without this assumption (and the proof is the same), except that then the solution need not be continuous.

For the proof of 1° we define the space F_1

$$F_1 = \{\varphi \in F : \varphi \text{ is increasing in } \langle 0, c \rangle\}.$$

Let us take $0\leqslant x_1 < x_2 \leqslant d$. From the monotonicity of $f(x)$ and from (2) we get

$$0\leqslant f(x_1) < f(x_2) \leqslant d.$$

For $\varphi \in F_1$ we have

$$0 \leqslant \varphi[f(x_1)] \leqslant \varphi[f(x_2)] \leqslant a.$$

Now from (6) and the monotonicity of $h(x, y)$ we obtain

$$\psi(x_1) = h(x_1, \varphi[f(x_1)]) \leqslant h(x_2, \varphi[f(x_1)]) \leqslant h(x_2, \varphi[f(x_2)]) = \psi(x_2),$$

i.e., ψ is increasing in $\langle 0, c\rangle$. Hence and from the first part of the proof it follows that $\psi \in F_1$. Since F_1 is a complete metric space, the unique solution φ continuous at $x=0$ and such that $\varphi(0)=0$ must belong to F_1. Thus φ is increasing in $\langle 0, c\rangle$. Now we shall prove that φ is increasing in $\langle 0, a)$. For this purpose we denote by x_0 the supremum of all b such that φ is monotonic in $\langle 0, b)$ and suppose that $x_0 < a$. From (2) we get $f(x_0) < x_0$. It follows from the continuity of $f(x)$ that there exists a number $\alpha > x_0$ such that for $x \in \langle x_0, \alpha)$ we have $f(x) < x_0$. Hence for $0 \leqslant x_1 < x_2 \leqslant \alpha$ we obtain

$$\varphi(x_1) = h(x_1, \varphi[f(x_1)]) \leqslant h(x_2, \varphi[f(x_2)]) = \varphi(x_2),$$

i.e., φ is monotonic in $\langle 0, \alpha\rangle$, $\alpha > x_0$. This contradiction completes the proof of the monotonicity of φ in $\langle 0, a)$.

If, for every fixed y, $h(x, y)$ is a strictly increasing function of the variable x, then for $0 \leqslant x_1 < x_2 < a$ we have

$$\varphi(x_1) = h(x_1, \varphi[f(x_1)]) < h(x_2, \varphi[f(x_1)]) \leqslant h(x_2, \varphi[f(x_2)]) = \varphi(x_2),$$

which proves that φ is strictly increasing.

REMARK 2. It is not sufficient to suppose that $h(x, y)$ is strictly increasing in the variable y. For instance, the unique solution continuous at $x=0$ of the equation

$$\varphi(x) = \tfrac{1}{2}\varphi[f(x)]$$

is $\varphi(x) \equiv 0$, and $h(x, y) = \frac{1}{2}y$ is strictly monotonic with respect to y.

For the proof of 2° we define the space F_2:

$$F_2 = \{\varphi \in F_1 : \varphi \text{ is convex in } \langle 0, c\rangle\}.$$

From the convexity and monotonicity of f, h, and φ we get for $0 \leqslant x_1 \leqslant c$, $0 \leqslant x_2 \leqslant c$ and ψ given by formula (6)

$$\psi\left(\frac{x_1+x_2}{2}\right) = h\left(\frac{x_1+x_2}{2}, \varphi\left[f\left(\frac{x_1+x_2}{2}\right)\right]\right) \leqslant h\left(\frac{x_1+x_2}{2}, \varphi\left[\frac{f(x_1)+f(x_2)}{2}\right]\right)$$

$$\leqslant h\left(\frac{x_1+x_2}{2}, \frac{\varphi[f(x_1)]+\varphi[f(x_2)]}{2}\right) \leqslant \frac{h(x_1, \varphi[f(x_1)]) + h(x_2, \varphi[f(x_2)])}{2} =$$

$$= \frac{\psi(x_1)+\psi(x_2)}{2}.$$

Hence $\psi(x)$ is convex in $\langle 0, c\rangle$. In view of the inclusion $F_2 \subset F_1$ and of the preceding part of the proof, (6) transforms F_2 into itself. Evidently, F_2 is a complete metric space, and thus the unique solution φ continuous at $x=0$ and such that $\varphi(0)=0$ must be convex. Similary as in preceding case we can prove that φ is convex in $\langle 0, a)$.

REMARK 3. The first part of the theorem is implicitly contained in the book [1] (comp. the remark at the end of § 4, Chapter III, p. 54).

REFERENCE

[1] M. KUCZMA: *Functional Equations in a Single Variable*, Monografie Matematyczne 46 (PWN, Warszawa 1968).

JANUSZ MATKOWSKI

O PEWNYCH WŁASNOŚCIACH ROZWIĄZAŃ RÓWNANIA FUNKCYJNEGO

Streszczenie

W pracy dowodzi się twierdzenia które orzeka, że jeżeli dane funkcje f i h są monotoniczne (względnie monotoniczne i wypukłe) to rozwiązanie równania (1) jest monotoniczne (wypukłe).

Oddano do Redakcji 1 sierpnia 1969 r.